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New Aspects in Physics on Gel'fand Triplets

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New observations in Gel'fand triplets are studied. An interesting one is vortex phenomena that stem from zero energy solutions of two-dimensional Schrödinger equations with central potentials $V(\rho) \propto \rho^r (\rho^2 = x^2 + y^2$ and $r \neq -2)$, which are eigenstates of conjugate spaces of Gel'fand triplets. The zero energy solutions for all the potentials $V(\rho)$ are shown to have the same structure with infinite degeneracy by making use of the conformal transformation $\xi^\alpha = z^\alpha$ with $z = x + iy$. The infinite degeneracy is observed as variety of vortex patterns in real physical phenomena. Some simple vortex patterns such as vortex lines and vortex lattices are presented. Such a new freedom on the Gel'fand triplets can be treated in a statistical mechanics. In the theory a new entropy being different from the so-called Boltzmann entropy appears. Transitions between the two entropies occur in thermal nonequilibrium phenomena, where energy emissions are observed.

KEY WORDS: Gel'fand triplets; vortex phenomena.

1. INTRODUCTION

It is well-known that conjugate spaces of Gel'fand triplets contain eigenstates with complex energy eigenvalues which describe resonances (Bohm and Gadella, 1989). It is also known that the complex eigenvalues appear in the pairs of complex conjugates, such as, $\epsilon \mp i\Gamma$ where $\epsilon \in \mathbb{R}$ and $\Gamma \in \mathbb{R}_+$ are, respectively, the energy and the decay width of resonances. The states with the $\mp$ sign, respectively, represent the resonance decay $(-)$ and formation $(+)$ processes. An example of such states for the parabolic potential barrier $V = -m\gamma^2 x^2/2$ in one dimension (1D PPB) has been studied by many authors (Balazs and Voras, 1990; Barton, 1986; Briet *et al.*, 1987; Castagnino *et al.*, 1997; Shimbori, 2000; Shimbori and Kobayshi, 2000a). It has been shown that the 1D PPB has pure imaginary energy eigenvalues $\mp i(n + 1/2)\hbar\gamma$ with $n = 0, 1, 2, \ldots$, and the eigenfunctions are generalized functions in the conjugate space $\mathcal{S}(\mathbb{R})^\times$ of Gel'fand triplet described by $\mathcal{S}(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^\times$, where $\mathcal{S}(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$, respectively, stand for a Schwartz space and a Lebesgue space (Castagnino *et al.*, 1997; Shimbori,

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2000; Shimbori and Kobayashi, 2000a). This means that all the states are unstable. We can, however, expect that stationary states will be constructed from those unstable states because the pairing property of the resonance decay and formation processes enable us to make states having zero imaginary energy by adding the two processes with the same probability. One can see an example of such cases in the two-dimensional parabolic potential barrier (2D PPB), that is, stationary states describing stationary flows appear in 2D PPB (Shimbori and Kobayashi, 200b). Actually those solutions are constructed from the pairs of the 1D PPB solutions with the energy eigenvalues $\mp i(n + 1/2)\hbar\gamma$ and then they are eigenstates having exactly zero energy, $\epsilon = 0$ and $\Gamma = 0$. As one can easily see, those zero energy solutions are infinitely degenerate, because the number of pairs corresponding to all positive integers n including 0 is an infinity. It is natural to ask a question whether this situation is a very peculiar property of 2D PPB. One will also ask another question how we can observe such infinitely degenerate zero energy solutions in real physical phenomena. In this article we shall show that this situation is a quite common property of models described by central potentials such as $V(\rho) \propto \rho^r$ for $r \neq -2$ in two dimensions, where $\rho^2 = x^2 + y^2$. And it is also pointed out that the infinite degeneracy can be observed as the infinite variety of vortex patterns. One may ask one more question how we can treat such a new freedom arising from the imaginary parts of energies in many body systems. A statistical mechanics on Gel'fand triplets will be presented and a new entropy corresponding to the freedom of the imaginary part of energies will be studied in the model (Kobayashi and Shimbori, 2000a,b; Kobayashi and Shimbori, 2001a).

In section 2 the common property of the zero energy solutions for central potentials is investigated in terms of conformal transformations. In section 3 the variety of vortex patterns originating from the infinite degeneracy of the zero energy solutions is discussed. In section 4 a statistical mechanics for the eigenstates with complex energies is studied and a new entropy for the imaginary energy freedom is discussed. Prospects of physics on Gel'fand triplets are briefly commented in section 5.

2. ZERO ENERGY SOLUTIONS OF TWO-DIMENSIONAL SCHRÖDINGER EQUATIONS WITH CENTRAL POTENTIALS

2.1. Conformal Transformations and Zero Energy Solutions

Let us start from the investigation of the eigenvalue problems of two-dimensional Schrödinger equations with the energy eigenvalue $\mathcal{E}$. The equations are written as

$$\left[-\frac{\hbar^2}{2m}\Delta + V_a(\rho)\right]\psi(x, y) = \mathcal{E}\psi(x, y), \tag{1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the central potentials are given by

$$V_a(\rho) = -a^2 g_a \rho^{2(a-1)},$$

with $\rho = \sqrt{x^2 + y^2}$, $a \in \mathbb{R}$ $(a \neq 0)$, and m and g_a are, respectively, the mass of the particle and the coupling constant. Note here that the eigenvalues $\mathcal{E}$ should generally be taken as complex numbers that are allowed in conjugate spaces of Gel'fand triplets (Bohm and Gadella, 1989). Note also that V_a represents repulsive potentials for $(g_a > 0, a > 1)$ and $(g_a < 0, a < 1)$ and attractive potentials for $(g_a > 0, a < 1)$ and $(g_a < 0, a > 1)$.

Here we consider the conformal mappings (Kobayashi, in press; Kobayashi and Shimbori, 2001b)

$$\zeta_a = z^a, \quad \text{with} \quad z = x + iy. \tag{2}$$

We use the notations u_a and v_a defined by $\zeta_a = u_a + iv_a$, where $u_a = \rho^a \cos a\varphi$ and $v_a = \rho^a \sin a\varphi$ with $\varphi = \arctan(y/x)$. In the (u_a, v_a) plane eq. (1) are written down as

$$a^2 \rho_a^{2(a-1)/a} \left[-\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = \mathcal{E} \psi(u_a, v_a), \tag{3}$$

where $\Delta_a = \partial^2/\partial u_a^2 + \partial^2/\partial v_a^2$. We can rewrite the equations as

$$\left[-\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = a^{-2} \mathcal{E} \rho_a^{2(1-a)/a} \psi(u_a, v_a). \tag{4}$$

Surprisingly the equations become same for all values of a except $a = 0$ when the energy eigenvalue $\mathcal{E}$ have exactly zero value. In other words, for $\mathcal{E} = 0$ the equations have the same form as that for the free particle with the constant potentials g_a as

$$[-\hbar^2 2m \Delta_a - g_a] \psi(u_a, v_a) = 0. \tag{5}$$

It should be noticed that in the case of $a = 1$ where the original potential is a constant g_1 the energy does not need to be zero but can take arbitrary real numbers, because the right-hand side of (4) has no ρ dependence. In the $a = 1$ case, therefore, we should take $g_1 + \mathcal{E}$ instead of g_a. Here let us briefly comment on the conformal mappings $\zeta_a = z^a$. We see that the transformation maps the part of the (x, y) plane described by $0 \leq \rho < \infty$, $0 < \varphi < \pi/|a|$ on the upper half-plane of the (u_a, v_a) plane for $a > 1$ and the lower half-plane for $a < -1$. Note here that the maps on the part of the (u_a, v_a) plane with the angle $\varphi_a = \varphi - \alpha$ can be carried out by using the conformal mappings

$$\zeta_a(\alpha) = z^a e^{-i\alpha}. \tag{6}$$

In the maps the variables are given by

$$u_a(\alpha) = u_a \cos\alpha + u_a \sin\alpha, \qquad v_a(\alpha) = v_a \cos\alpha - u_a \sin\alpha. \tag{7}$$

The relations $u_a(0) = u_a$ and $v_a(0) = v_a$ are obvious. Note that the angle α represents the freedom of the angle of the incoming stationary wave.

It is trivial that the Eq. (5) for all a have the particular solutions

$$\psi_0^{\pm}(u_a) = N_a\, e^{\pm i k_a u_a}, \qquad \psi_0^{\pm}(v_a) = N_a\, e^{\pm i k_a v_a}, \qquad \text{for } g_a > 0 \tag{8}$$

and

$$\phi_0^{\pm}(u_a) = M_a\, e^{\pm k_a u_a}, \qquad \phi_0^{\pm}(v_a) = M_a\, e^{\pm k_a v_a}, \qquad \text{for } g_a > 0 \tag{9}$$

where $k_a = \sqrt{2m|g_a|}/\hbar$ and N_a and M_a are in general complex numbers. General solutions should be written by the linear combinations of (8) for $g_a > 0$ and those of (9) for $g_a < 0$. Examples for the 2D PPB with $g_a > 0$ are presented in Shimbori and Kobayashi; (2000b). In the following investigations we shall concentrate our attention on the solutions of (8) that are expressed in terms of plane waves. Hereafter we use the notations $u_a(\alpha)$ and $v_a(\alpha)$ with $-\pi < \alpha \leq \pi$, which are used in the conformal mappings of (6). We see that

$$e^{\pm i k_a u_a(\alpha)} \quad \text{and} \quad e^{\pm i k_a v_a(\alpha)}$$

are the solutions of (5). (For details, see Kobayashi and Shimbori, 2001b.)

2.2. Infinite Degeneracy of the Zero Energy Solutions

Let us study the degeneracy of the solutions. The origin of the infinite degeneracy can easily be understood in the case of the 2D PPB (Shimbori and Kobayashi, 2000b). It is known that energy eigenvalues of 1D PPB are given by pure imaginary values $\mp i(n + 1/2)\hbar\gamma$ with $n = 0, 1, 2, \ldots$ (Balazs and Voras, 1990; Barton, 1986; Briet *et al.*, 1987; Castagnino *et al.*, 1997; Shimbori, 2000; Shimbori and Kobayshi, 2000a). From this result we see that the energy eigenvalues of the 2D PPB, which are composed of the sum of the 1D PPB eigenvalues with the opposite signs such as $\mp i(n_x - n_y)\,\hbar\gamma$ with n_x and $n_y = 0, 1, 2, \ldots$, include the zero energy for $n_x = n_y$. It is apparent that all the states with the energy eigenvalues $\mp i(n_x - n_y)\,\hbar\gamma$ are infinitely degenerate (Shimbori and Kobayashi, 2000b). This means that all the central potentials $V_a(\rho)$ have the same degeneracy for the zero energy states. By putting the wave function $f^{\pm}(u_a; v_a)\psi_0^{\pm}(u_a)$ into (5) where $f^{\pm}(u_a; v_a)$ are polynomial functions of u_a and v_a, we obtain the equation

$$\left[\Delta_a \pm 2ik_a \frac{\partial}{\partial u_a}\right] f^{\pm}(u_a; v_a) = 0. \tag{10}$$

A few examples of the functions f are given by (Shimbori and Kobayashi, 2000b)

$$\begin{aligned} f_0^{\pm}(u_a; v_a) &= 1, \\ f_1^{\pm}(u_a; v_a) &= 4k_a v_a, \\ f_2^{\pm}(u_a; v_a) &= 4\left(4k_a^2 v_a^2 + 1 \pm 4ik_a v_a\right). \end{aligned} \tag{11}$$

We can obtain the general forms of the polynomials in the 2D PPB, which are generally written by the multiple of the polynomials of degree n, $H_n^{\pm}(\sqrt{2k_2}x)$, such that

$$f_n^{\pm}(u_2; v_2) = H_n^{\pm}\left(\sqrt{2k_2}x\right) \cdot H_n^{\pm}\left(\sqrt{2k_2}y\right), \tag{12}$$

where x and y in the right-hand side should be considered as the functions of u_2 and v_2 (Shimbori and Kobayashi, 2000b). Since the form of the Eq. (5) is the same for all a, the solutions can be written by the same polynomial functions that are given in (12) for the PPB. That is to say, we can obtain the polynomials for arbitrary a by replacing u_2 and v_2 with u_a and v_a in (12). Note that the polynomials $H_n^{\pm}(\xi)$ with $\xi = \sqrt{m\gamma/\hbar}x$ are defined by the solutions for the eigenstates with $\mathcal{E} = \mp i(n + 1/2)\hbar\gamma$ in 1D PPB of the type $V(x) = -m\gamma^2 x^2/2$ and they are written in terms of the Hermite polynomials $H_n(\xi)$ as

$$H_n^{\pm}(\xi) = e^{\pm in\pi/4} H_n(e^{\mp i\pi/4}\xi). \tag{13}$$

(For details, see Castagnino *et al.*, 1997; Shimbori, and Kobayashi, 2000a). Note here that these wave functions in the two dimensions are the generalized functions of the conjugate spaces $\mathcal{S}(\mathbb{R}^2)^{\times}$ in Gel'fand triplets, of which nuclear space is given by Schwarz space $\mathcal{S}(\mathbb{R}^2)$ (a linear subspace of Lebesgue space $\mathcal{L}^2(\mathbb{R}^2)$), such that $\mathcal{S}(\mathbb{R}^2) \subset \mathcal{L}^2(\mathbb{R}^2) \subset S(\mathbb{R}^2)^{\times}$. (For details, see Bohm and Gadella, 1989).

The extension to three dimensions can easily be carried out in the cases with potentials that are separable into the (x, y) plane and the z direction such that $V(x, y, z) = V_a(\rho) + V(z)$. When the energy eigenvalues of the z direction are given by E_z, we obtain the same equation as (6) for $\mathcal{E} - E_z = 0$. (For the $a = 1$ case $g_1 + \mathcal{E} - E_z > 0$ should be taken.) If we take the free motion with the momentum p_z for the z direction, $E_z = P_z^2/2m$ should be taken. It is important that the total energy $\mathcal{E}$ is in general not equal to zero in the three dimensions. Note that wave functions for the separable potentials are written by the product such as $\psi(x, y, z) = \psi(x, y)\psi(z)$. Hereafter we shall not explicitly write $\psi(z)$ in the wave functions.

3. VORTICES IN ZERO ENERGY SOLUTIONS

3.1. Hydrodynamical Approach and Vortices

The vortices that are well-known objects in hydrodynamics have been investigated in many aspects (Batchelor, 1967; Lamb, 1932; Landau and Lifshitz, 1987; saffman, 1992). In quantum mechanics, hydrodynamical approach was vigorously investigated in the early stage of the development of quantum mechanics (Bohm, 1952; Bohm and Vigier, 1954; De Broglie, 1930; Dirac, 1951; Kennard, 1928; Madelung, 1926; Schönberg, 1954; Takabayasi, 1952).

The fundamental properties of vortices in quantum mechanics were extensively examined by Hirschfelder and others (Ghosh, 1982; Hirschfelder, 1977; Hirschfelder *et al.*, 1974a,b; Hirschfelder and Tang, 1976) and the motions of vortex lines were also studied (Bialynicki-Biruk *et al.*, 2000; Schecter and Dubin, 1999).

Let us study vortices that appear in the linear combinations composed of the solutions with the polynomials of (12). Before going into the details we briefly describe vortices in quantum mechanical hydrodynamics. The probability density $\rho(t, x, y)$ and the probability current $j(t, x, y)$ of a wave function $\psi(t, x, y)$ in nonrelativistic quantum mechanics are defined by

$$\rho(t, x, y) \equiv |\psi(t, x, y)|^2, \tag{14}$$

$$j(t, x, y) \equiv \mathrm{Re}[\psi(t, x, y)^*(-i\hbar\nabla)\psi(t, x, y)]/m. \tag{15}$$

They satisfy the equation of continuity $\partial\rho/\partial t + \nabla \cdot \boldsymbol{j} = 0$. Following the analogue of the hydrodynamical approach (De Broglie, 1930; Dirac, 1951; Kennard, 1928; Madelung, 1926; Saffman, 1992), the fluid can be represented by the density ρ and the fluid velocity v. They satisfy Euler's equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho v) = 0. \tag{16}$$

Comparing this equation with the continuity equation, the following definition for the quantum velocity of the state $\psi(t, x, y)$ is led in the hydrodynamical approach:

$$v \equiv \frac{\boldsymbol{j}(t, x, y)}{|\psi(t, x, y)|^2}. \tag{17}$$

Notice that ρ and $\boldsymbol{j}$ in the present cases do not depend on time t. Now it is obvious that vortices appear at the zero points of the density, that is, the nodal points of the wave function. At the vortices, of course, the current $\boldsymbol{J}$ must not vanish. We should here remember that the solutions of (5) degenerate infinitely. This fact indicates that we can construct wave functions having the nodal points at arbitrary positions in terms of linear combinations of the infinitely degenerate solutions (Kobayashi and Shimbori, 2001b; Shimbori and Kobayashi, 2000b).

The strength of vortex is characterized by the circulation Γ that is represented by the integral round a closed contour C encircling the vortex such that

$$\Gamma = \oint_c v \cdot ds \tag{18}$$

and it is quantized as

$$\Gamma = 2\pi l\hbar/m, \tag{19}$$

where the circulation number l is an integer (Bialynicki-Biruk *et al.*, 2000; Hirschfelder, 1977; Hirschfelder *et al.*, 1974b). It should be stressed that we can perform the investigation of vortices for all the cases except $a = 0$ in the (u_a, v_a) plane, because fundamental properties of vortices such as the numbers of vortices in the original plane and the mapped plane and the strengths of vortices do not change by the conformal mappings.

3.2. Vortex Patterns

Vortex patterns in the (x,y) plane can be obtained by the inverse transformations of the conformal mappings. Let us here show that vortex lines and vortex lattices can be constructed from simple linear combinations of the low lying polynomial solutions. And also, the mapped patterns of those lines and lattices are investigated by the conformal mappings for $a = 2$ (PPB case; $V_a \propto \rho^2$) and for $a = 1/2$ (Coulomb type; $V_a \propto \rho^{-1}$). In the following discussions the suffices a of u_a, v_a, and k_a are omitted.

Vortex Lines

Let us consider the linear combination of 2° one solutions such that

$$\Psi(u, v) = ve^{iku} - ue^{-ikv}, \tag{20}$$

where the complex constant corresponding to the overall factor of the wave function is ignored, because the wave function belongs to the conjugate space of Gel'fand triplet and is not normalizable. This means that the wave function represents a stationary flow such as in scattering processes. The nodal points of the probability density $|\Psi(u, v)|^2 = u^2 + v^2 - 2uv\cos k(u + v)$ appear at points satisfying the conditions

$$u = \pm v, \qquad \cos k(u + v) = \pm 1. \tag{21}$$

We have the nodal points at

$$u = v = n\pi/k, \quad \text{for} \quad n = \text{integers}. \tag{22}$$

In the (u, v) plane the positions of vortices can be on a line of $u = v$. After some elementary but tedious calculations, we see that the circulation numbers of vortex

strengths are given by $l = -1$ for $n =$ positive integers and $l = 1$ for $n =$ negative ones. Note that the origin at $u = v = 0$ has no vortex. We can directly see the result by showing the fact that the strength of vortex Γ becomes zero for the closed circle around the origin. We can interpret this result as follows: at the origin there exist a pair of vortices having the opposite circulation numbers, that is, they, respectively, belong to the vortex line with $l = -1$ and that with $l = 1$. We may say that it is a vortex dipole.

For the case of $a = 1$ (constant potential) we can take as $u = x$ and $v = y$.

In the case of $a = 2$ (PPB) we have $u = x^2 - y^2$ and $v = 2xy$. The relations for the nodal points are written down as

$$y = \frac{1}{\sqrt{2}+1}x, \qquad y = \pm\frac{1}{2}\sqrt{\frac{n\pi}{(\sqrt{2}+1)k}}, \quad \text{for } n = \text{ positive integers}$$

$$y = -\frac{1}{\sqrt{2}-1}x, \qquad y = \pm\frac{1}{2}\sqrt{\frac{|n|\pi}{(\sqrt{2}-1)k}}, \quad \text{for } n = \text{negative integers.} \tag{23}$$

We see that a vortex quadrupole composed of two vortex dipole appears at the origin.

In the case of $a = 1/2$ (Coulomb-type), by using the relations $u^2 - v^2 = x$ and $2uv = y$, we obtain the conditions for the nodal points as follows:

$$x = 0, \qquad y = 2\frac{n^2\pi^2}{k^2}, \quad \text{for } n = \text{nonzero integers.} \tag{24}$$

Note that the origin is a singular point, where the source of the potential exists.

Figures for $a = 1$, 2, and 1/2 are presented in Figs. 1, 2, and 3, which, respectively, represent the vortex pattern for the constant potential, that for the PPB, and that for the Coulomb type one.

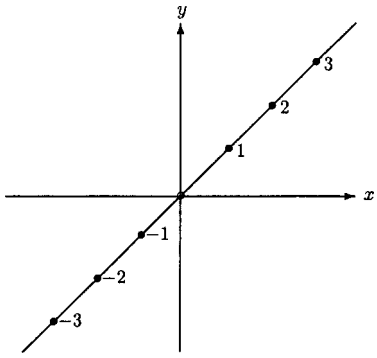

Fig. 1. Positions of vortices for $n = \pm 1, \pm 2, \pm 3$ in the constant potential ($a = 1$), which are denoted by $\bullet$.

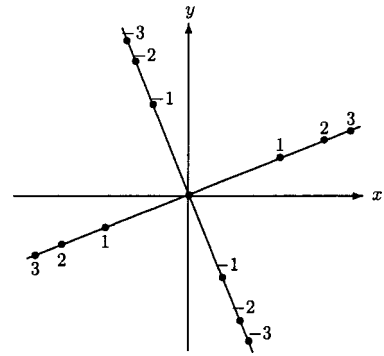

Fig. 2. Positions of vortices for $n = \pm 1, \pm 2$, and ± 3 in the PPB ($a = 2$), which are denoted by •.

Note here that the differences of the potentials clearly appear not only in the vortex patterns but the properties of the singularities at the origin as well. Also notice that the parallel vortex lines are constructed from the linear combinations of the lowest and the degree 1 polynomials (Kobayashi and Shimbori, 2001b).

Vortex Lattices

Let us consider the linear combination of a stationary wave and a plane wave such that

$$\Psi(u, v) = \cos ku - e^{-ikv}. \tag{25}$$

The nodal points of the probability density $|\Psi(u, v)|^2 = 1 + \cos^2 ku - 2\cos ku \cos kv$ appear at positions satisfying

$$u = m\pi/k, \qquad v = n\pi/k, \tag{26}$$

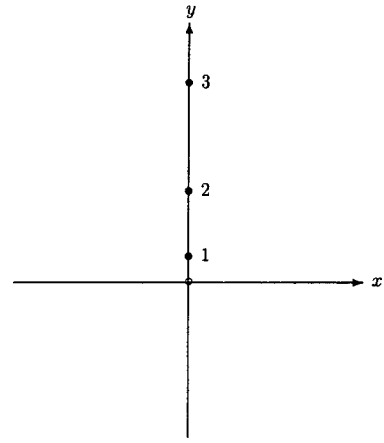

Fig. 3. Positions of vortices for $n = 1, 2$, and 3 in the Coulomb type-potential ($a = 1/2$), which are denoted by •.

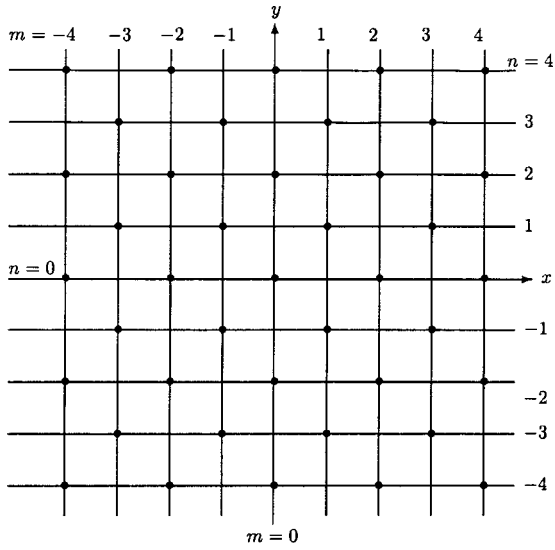

Fig. 4. Positions of vortices for $m, n = 0, \pm 1, \pm 2, \pm 3$, and ± 4 in a constant potential ($a = 1$), which are denoted by • and the distance between the neighboring lines are taken by π/k.

where both of m and n must be even or odd, that is, $(-1)^m = (-1)^n$. These conditions produce a vortex lattice presented in Fig. 4 which was suggested in Kobayashi and Shimbori, (2000a).

In the cases of the PPB ($a = 2$) and the Coulomb, type ($a = 1/2$) vortices appear at the cross points of the following two functions:

$$\begin{aligned} &x^2 - y^2 = m\pi/k, \qquad xy = n\pi/2k, \quad \text{for the PPB,} \\ &x^2 + y^2 = (m^2 + n^2)^2\pi^4/k^4, \qquad y = 2mn\pi^2/k^2, \quad \text{for the Coulomb type.} \end{aligned} \tag{27}$$

In the arbitrary values of a we obtain the circulation number $l = -1$ for the all vortices. Figures for $a = 2$ and 1/2 are given in Figs. 5 and 6, respectively.

In these arguments we see the following points:

(1) The construction of vortex lattices in experiments seems to be not very difficult. In fact the vortex lattice of (ii) can be produced from a stationary wave and a plane wave perpendicular to the stationary wave.
(2) The differences of potentials can be clearly seen from the vortex patterns. Especially the distances between two neighboring vortices are a good object to identify the type of the potentials. That is to say, the vortices

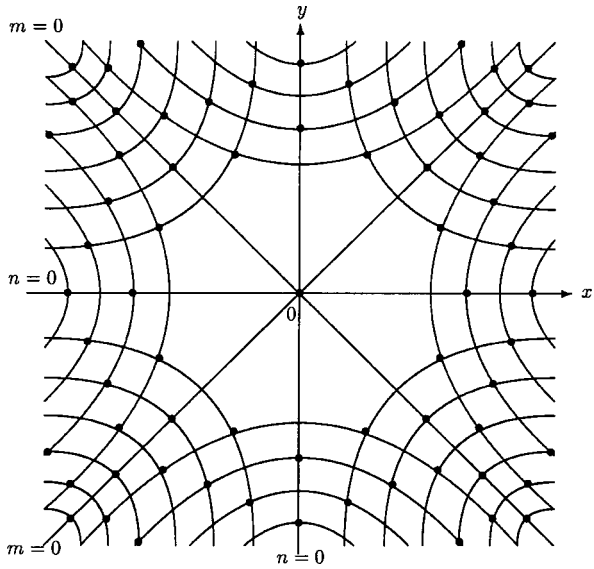

Fig. 5. Positions of vortices for the PPB ($a = 2$), which are denoted by •.

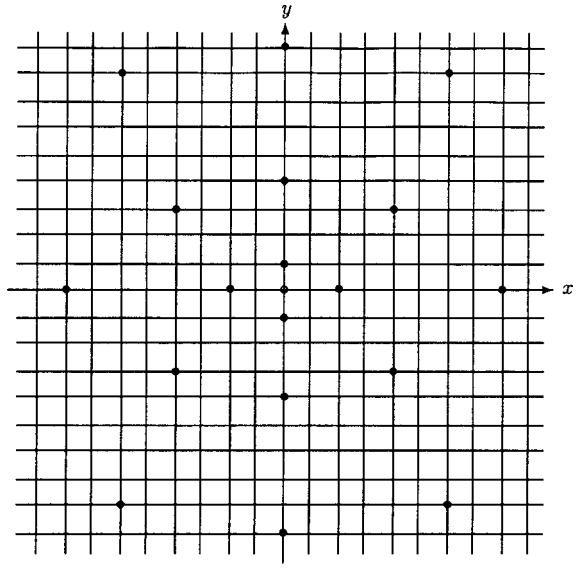

Fig. 6. Positions of vortices for $0 \leq |m|, |n| \leq 3$ in the Coulomb-type potential ($a = 1/2$), which are denoted by •, the distance between the neighboring lines are taken by $2(\pi/k)^2$.

appear in an equal distance π/k in the case of the constant potential ($a = 1$), whereas the distances become smaller in the regions far from the origin for $a > 1$ and larger for $a < 1$ in comparison with those near the origin.

(3) The property of the singularity at the origin is also a good object to identify the potentials.

Although it is at this moment difficult to categorize the present experimental vortex patterns (Batchelor, 1967; Kobayashi, in press; Kobayashi and Shimbori, 2000a,b; Kobayashi and Shimbori, 2001a,b; Lamb, 1932; Landau and Lifshitz, 1987 Shimbori and Kobayashi, 2000b), we shall be able to understand fundamental dynamics of vortex phenomea from vortex patterns.

It is also noticed that the present results can be applicable not only to quantum phenomena but also those in classical fluids by changing the parameters m, $\hbar$, and g_a in the original equation.

Uphil now we have not discussed the stability of the vortex lattices. To investigate the time development of the patterns we have to take account of the fact that the solutions used here belong to the conjugate spaces of Gel'fand triplets. In the spaces the eigenstates generally have complex energy eigenvalues which are expressed by the pairs of complex conjugates corresponding to the resonance decay and formation processes. We see that this pairing property is the origin of the infinite degeneracy of the solutions and the infinite degeneracy stems from the balance between the resonance decay and formation processes. This fact seems to indicate that vortex systems are possibly unstable for perturbations. Actually the existence of vortex lattices has already been pointed out, and it has also been noticed that those systems will decay from their edges, where the balance between the decay and formation processes is broken (Kobayashi and Shimbori, 2000a). This problem will be discussed in the final section again.

4. STATISTICAL MECHANICS ON GEL'FAND TRIPLETS AND ENTROPIES

4.1. Statistical Mechanics on Gel'fand Triplets

We have seen that the imaginary energy freedom in Gel'fand triplets brings very interesting physical situations, where even stationary states can be included. Now we should also study what the imaginary energy freedom brings in many body systems. We, therefore, construct a statistical mechanics in which the imaginary energy freedom is introduced. Let us start from the construction of

microcanonical ensemble for states having complex energies which are generally represented by

$$\varepsilon_i \, j_i = \epsilon_i - i\gamma_{j_i} \quad \text{for } \epsilon_i, \gamma_{j_i} \in \mathbb{R}, \tag{28}$$

where i, $j_i \in \{0, 1, 2, \ldots\}$ and the suffix i of j_i is needed when there is some relation between the real and imaginary energy eigenvalues. We consider a simple case described by a system composed of N independent particles being in complex energy states. In this case the total energy of the N-particle system is given by the sum of energy eigenvalues of each particles such that

$$\mathcal{E} = E - i\Gamma, \tag{29}$$

where

$$E = \sum_i \varepsilon_i \quad \text{and} \quad \Gamma = \sum_{j_i} \gamma_{j_i}. \tag{30}$$

Note here that the absolute value of the total imaginary energy $|\Gamma|$ can be taken as small as possible because of the pairing property of all complex energy eigenvalues in Gel'fand triplets. This means that systems having small $|\Gamma|$ can be stable enough to realize a thermal equilibrium which is described in statistical mechanics. Here we shall investigate simple cases where the real and imaginary energy eigenvalues are independently determined and then we can take off the suffix i from j_i. Such models will explicitly be presented in cases with parabolic potential barriers. The basic principle is taken as same as that for the usual statistical mechanics, that is, principle of equal a priori probability. Then we start from counting the number of independent combinations of states for a fixed energy $\mathcal{E}$. Since two freedoms concerning to the real and imaginary parts of energies are independent of each other, the number of the combination (thermodynamical weight) $W(\mathcal{E})$ is counted by the product of the number $W^{\mathfrak{R}}(E)$ for realizing the real part E and that $W^{\mathfrak{F}}(\Gamma)$ for realizing the imaginary part Γ such that

$$W(\mathcal{E}) = W^{\mathfrak{R}}(E) W^{\mathfrak{F}}(\Gamma). \tag{31}$$

Following the procedure of statistical mechanics, we now see that the entropy $S(\mathcal{E}) = k_B \log W(\mathcal{E})$ of the system is written in terms of the sum of two entropies such that

$$S(\mathcal{E}) = S^{\mathfrak{R}}(E) S^{\mathfrak{F}}(\Gamma), \tag{32}$$

where k_{B} is the Boltzmann constant and $S^{\mathfrak{R}}(E) = K_{\mathrm{B}} \log W^{\mathfrak{R}}(E)$ and $S^{\mathfrak{F}}(\Gamma) = K_{\mathrm{B}} \log W^{\mathfrak{F}}(\Gamma)$ are, respectively, the Boltzmann entropy and the new entropy induced from the freedom of the imaginary part.

Let us consider equilibrium between two systems which can transfer only energies with each other. The total energy $\mathcal{E} = E - i\Gamma$ given by the sum of those

for two systems $\mathcal{E}_{\rm I} = E_{\rm I} - i\Gamma_{\rm I}$ and $\mathcal{E}_{\rm II} = E_{\rm II} - i\Gamma_{\rm II}$ is fixed. The number of the available combinations is written by the product of those for the two systems as

$$W(\mathcal{E}) = W_{\rm I}(\mathcal{E}_{\rm I})W_{\rm II}(\mathcal{E}_{\rm II}), \tag{33}$$

where $W_{\rm I}(\mathcal{E}_{\rm I}) = W_{\rm I}^{\mathfrak{R}}(E_{\rm I})W_{\rm I}^{\mathfrak{F}}(\Gamma_{\rm I}$ and $W_{\rm II}(\mathcal{E}_{\rm II}) = W_{\rm II}^{\mathfrak{R}}(E_{\rm II})W_{\rm II}^{\mathfrak{F}}(\Gamma_{\rm II})$. Now we have the entropy expressed as the sum of four terms

$$S(\mathcal{E}) = S_{\rm I}^{\mathfrak{R}}(E_{\rm I}) + S_{\rm I}^{\mathfrak{F}}(\Gamma_{\rm I}) + S_{\rm II}^{\mathfrak{R}}(E_{\rm II}) + S_{\rm II}^{\mathfrak{F}}(\Gamma_{\rm II}), \tag{34}$$

where $S_{\rm I}^{\mathfrak{R}}(E_{\rm I}) = k_{\rm B}\log W_{\rm I}^{\mathfrak{R}}(E_{\rm I})$ and so on. In the procedure maximizing the entropy $S(\mathcal{E})$ under the constraints that $E = E_{\rm I} + E_{\rm II}$ and $\Gamma = \Gamma_{\rm I} + \Gamma_{\rm II}$ are fixed, we obtain two independent relations corresponding to the two constraints such that

$$\frac{\partial S_{\rm I}^{\mathfrak{R}}(E_{\rm I})}{\partial E_{\rm I}} = \frac{\partial S_{\rm II}^{\mathfrak{R}}(E_{\rm II})}{\partial E_{\rm II}}, \qquad \frac{\partial S_{\rm I}^{\mathfrak{F}}(\Gamma_{\rm I})}{\partial \Gamma_{\rm I}} = \frac{\partial S_{\rm II}^{\mathfrak{F}}(\Gamma_{\rm II})}{\partial \Gamma_{\rm II}}. \tag{35}$$

The first relation leads the usual temperature but the second one produces a new quatity which must be same for the two systems in equilibrium. The canonical distribution for the energy $\mathcal{E}_{lm} = E_l - i\Gamma_m$ is written by

$$P(\mathcal{E}_{lm}) = Z^{-1}\exp\left(-\beta^{\mathfrak{R}}E_l - \beta^{\mathfrak{F}}\Gamma_m\right), \tag{36}$$

where $\beta^{\mathfrak{R}}$ should be chosen as the usual factor $\beta = (K_{\rm B}T)^{-1}$ of canonical distribution, $\beta^{\mathfrak{F}}$ denotes the new physical quantity in the equilibriums and the canonical partition function Z is given by

$$Z = \sum_t \sum_m \exp\left(-\beta E_l - \beta^{\mathfrak{F}}\Gamma_m\right).$$

What is the new quantity $\beta^{\mathfrak{F}}$? In independent particle systems wave functions are written by the product of all constituents such that

$$\Psi(t, r_1, \ldots, r_N|\mathcal{E}) = \prod_{n=1}^{N}\psi(t, r_n|\varepsilon_n), \tag{37}$$

where the wave function for one constituent with $\varepsilon_n = \varepsilon_n - i\gamma_n$ is generally given by $\psi(t, r_n|\varepsilon_n) = e^{-i\varepsilon_n t/\hbar}\phi(r_n)$. The probability density for Ψ at the time t is evaluated as

$$\begin{aligned}\rho(t, r_1, \ldots, r_N|\mathcal{E}) &= |\Psi(t, r_1, \ldots, r_N|\mathcal{E})|^2 \\ &= e^{-2\Gamma t/\hbar}\prod_n |\phi(r_n)|^2,\end{aligned} \tag{38}$$

where $\Gamma = \sum_n \gamma_n$. We see that all the states with the same total imaginary energy Γ have the same time dependence $e^{-2\Gamma t/\hbar}$. Since the states with complex energy eigenvalues are unstable, the canonical distribution $P(\mathcal{E})$ must depend on time.

It is natural that the time dependence of $P(\mathcal{E})$ is same as that of the probability density, which is determined by the imaginary part Γ of the total energy $\mathcal{E}$ of the system. We can specify

$$\beta^{\widetilde{\mathfrak{F}}} = 2t/\hbar. \tag{39}$$

Thus, we can introduce a common time scale t for two systems being in equilibriums. Note that the imaginary parts γ_j are expressed by pairs of conjugates, that is, $\pm|\gamma_j|$ $(\forall j \in Z_+)$. This fact means that the total imaginary part Γ can possibly be in microscopic order (quantum size), even if the total real part E is in macroscopic order.

It is interesting that in the present equilibrium all constituents are governed by one common time scale. This situation seems to be very interesting to describe the Universe with one time scale.

The introduction of free energies for the real and imaginary freedoms are straightforward. We can make grand canonical ensemble, where chemical potentials having the time (common time) dependence are obtained. (For details, see Kobayashi and Shimbori, 2001a).

4.2. Entropy Transfer From $S^{\widetilde{\mathfrak{F}}}$ to $S^{\widetilde{\mathfrak{R}}}$

Let us study the entropy transfer from $S^{\widetilde{\mathfrak{F}}}$ to $S^{\widetilde{\mathfrak{R}}}$ in an adiabatic process described by a decay of a system that is composed of N resonances in a 1D PPB+ some ordinary potentials, where the ordinary potentials mean potentials which are described by Hilbert spaces. In this model, therefore, $S^{\widetilde{\mathfrak{F}}}$ and $S^{\widetilde{\mathfrak{R}}}$, respectively, stand for the entropy of the PPB system and that of the ordinary system. Here we study the process where the decays of the resonance system are absorbed into the system described by the ordinary potentials. (For details, see Kobayashi and Shimbori, 2000b; Kobayashi and Shimbori, 2001a.) After the decay processes are opened at $t = 0$, the entropy of the system being in the PPB is obtained as (Kobayashi and Shimbori, 2001a)

$$S^{\widetilde{\mathfrak{F}}} = Nk_{\mathrm{B}}\left[2\gamma t\frac{e^{2\gamma t}}{e^{2\gamma t}-1} - \log(e^{2\gamma t}-1)\right]. \tag{40}$$

For small t such that $\gamma^t \ll 1/2$ the entropy behaves

$$S^{\widetilde{\mathfrak{F}}} \simeq -Nk_{\mathrm{B}}\log\tau \tag{41}$$

where $\tau = \gamma t$. This relation gives us

$$dS^{\widetilde{\mathfrak{F}}} - Nk_{\mathrm{B}}\frac{d\tau}{\tau} \quad \text{for } \tau \ll 1/2. \tag{42}$$

Since the total entropy conserves in the adiabatic process, that is, the relation

$$dS = dS^{\mathfrak{R}} + dS^{\mathfrak{F}} = 0 \tag{43}$$

holds, we obtain the relation

$$dS^{\mathfrak{R}} = -dS^{\mathfrak{F}}. \tag{44}$$

Note here that $dS^{\mathfrak{R}}$ is always positive because $dS^{\mathfrak{F}} < 0$ is kept. In the system described only by PPBs the temperature T originated from the freedom of real energy eigenvalues is zero, that is $T = 0$, since the system has no real energy freedom. This means that the temperature must be zero at $t = 0$, that is, just at the moment when the decay processes are opened. Let us write it as

$$T(t) = K_0 \tau^{\delta} \quad \text{for } \tau \ll 1/2. \tag{45}$$

where K_0 and δ should be positive constants. Since the direct observable in this process is the real energy $E^{\mathfrak{R}}$ released into the ordinary potentials by the decay of resonances, we should evaluate the real energy produced in this process. For the small t we have

$$dE^{\mathfrak{R}} = T(t)\, dS^{\mathfrak{R}} = N k_{\mathrm{B}} K_0 \tau^{\delta-1}\, d\tau \quad \text{for } \tau \ll 1/2. \tag{46}$$

Then we can estimate the real energy produced in the process during the short period from 0 to t ($\ll 1/2\gamma$) as

$$E^{\mathfrak{R}} = \int_0^{\gamma t} \frac{dE^{\mathfrak{R}}}{d\tau}\, d\tau = N k_{\mathrm{B}} \frac{K_0}{\delta} (\gamma t)^{\delta}. \tag{47}$$

Since $\delta > 0$, this process produces a real positive energy. The unknown constants K_0 and δ will depend on the property of the system where the produced energy is absorbed. We see that the system in PPBs can be the source of the energy production. It, of course, does not mean the breakdown of the energy conservation law. In the process where the system is composed in the PPB the real energy produced in the decay process is stored as $S^{\mathfrak{F}}$ in the system. This means that the total produced energy which is evaluated by the integration from $t = 0$ to ∞ must coincide with the energy consumed in the process for making the initial system. This integration will derive a relation between K_0 and δ. An example of energy production was discussed in Kobayashi and Shimbori (2000b).

5. PROSPECTS OF PHYSICS ON GEL'FAND TRIPLETS

We have shown that Schrödinger equations with central potentials in two dimensions have common zero energy solutions being degenerate infinitely and those degeneracy are observed as the very rich variety of vortex phenomena. Actually problems of vortices appear in many aspects of present-day physics such as

vortex matters (vortex lattices) (Blatter, 1994; Crabtree and Nelson, 1997), vortices in nonneutral plasma (Fine *et al.*, 1995; Lto *et al.*, 2001; Kiwamoto, 1999, 2000a), and Bose–Einstein gases (Fitzlerland *et al.*, 2000; Madison *et al.*, 2000; Marago *et al.*, 2000; Mathews *et al.*, 1999; Raman *et al.*, 1999) and so on. Although the relations between the present model and those observed processes are not yet clear, the study of vortex phenomena will open a new prospect in physics based on Gel'fand triplets. It should be remarked that the huge degeneracy of the zero energy solutions also provide the huge degeneracy of stable states including vacuum, because the addition of the zero energy states does not change the total energy at all. We can understand such systems as follows: Inside of the systems both of the decay and formation processes of unstable states like resonances always occur with the same probability and then the total systems can be stable. An example was presented as two-dimensional lattices connected by stationry flows that are described by the zero energy solutions (Kobayashi, and Shimbori, 2000). (see Fig. 7.) These systems have two interesting properties. One is the fact that inside of the systems every two lattice points are connected by stationary flows which are observable in quantum mechanics. We may, therefore, call those states composed of observable quantities semiclassical states. It seems to be very attractive to study mesoscopic phenomena being in broders between quantum and classical phenomena in terms of the present scheme. The other point is that, as seen in Fig. 7, such lattices can be unstable at the edges of the lattices, because the systems touch outer environments at the edges and then the decay and formation processes are in general no more in

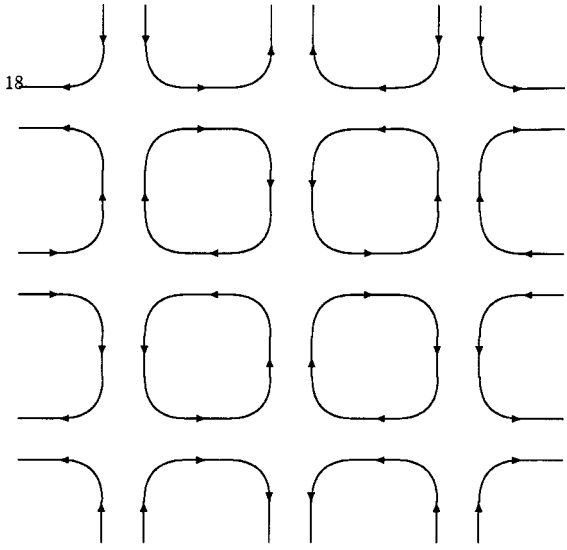

Fig. 7. Two-dimensional lattice connected by stationary flows.

balance at the edges. Thus the systems will slowly decay at their edges. Actually we quite often observe such matters decaying from their edges including their surfaces in our daily life. To investigate those complex matters we have clearly to understand the new freedom for the imaginary energy eigenvalues which are quite common observations in Gel'fand triplets. Though statistical mechanics on Gel'fand triplets presented here show us some ideas such as the new entropy for the imaginary energy freedom, the theory is still too primitive to investigate realistic phenomena including slowly decaying or changing matters. it should, however, be noted that the stable lattices are possibly constructed on closed surfaces like balls and torus. Provided that a macroscopic stable lattice is made, we can observe a macroscopic energy emission by breaking its stability by some perturbations. Such energy emissions without nuclear fusions will possibly be observed even at ordinary temperatures as very peculiar phenomena which are very hard to be understood in dynamics on Hilbert spaces. It is also an interesting idea to draw the birth of the Universe in terms of the collision of two huge stable-lattices, where the common time scale describing the universe can be naturally introduced as shown in §4.1. We may say that the investigation on the freedom corresponding to the imaginary part of energy that is essentially a new object in Gel'fand triplets seems to be very promising theme in the present-day physics.

Finally we briefly mention other problems which are not touched here. A supersymmetric theory for scattering can be realized on Gel'fand triplets (Shimbori and Kobayashi, 2001). It should also be noted that the pure imaginary energy eigenvalues such as those in PPB possibly provide tachyons in relativistic dynamics and those tachyons can be states describing Higgs mechanisms for spontaneous symmetry breaking of vacuum.

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